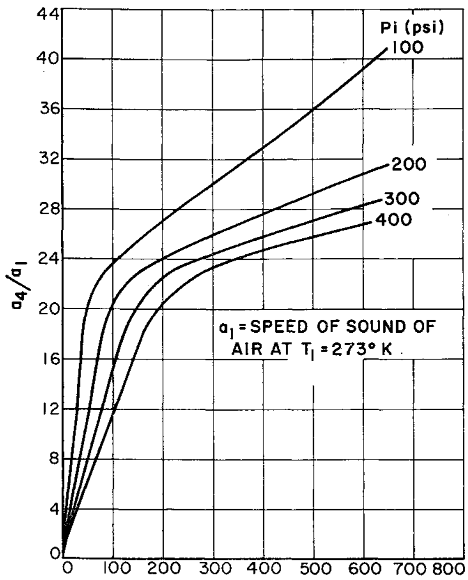


**Fig. 1 Speed of sound ratio for energy input to a constant volume of helium**

development of a hypervelocity research facility at Jet Propulsion Laboratory, it was felt that more accurate calculations were needed. The recent publication at Harvard[3] of a Mollier diagram for helium has made it possible to calculate the real gas properties of the helium driver as a function of energy input. Some extrapolation of the Harvard results was needed in order to complete the computation.

These calculations should make it easier to predict more accurately the performance of hypervelocity shock tubes. Figures 1–3 give the final temperature and pressure and the sound speed ratio to air of the driver as a function of the initial conditions and the energy input per unit volume. The initial energy of the gas has been neglected.

### References

[1] Rose, P. H. and Camm, J. C., "Electric shock tube for high velocity simulation," Avco-Everett Res. Lab., Res. Rept. 136 (July 1962).

[2] Warren, W. R., Rogers, D. A., and Harris, C. J., "The development of an electrically heated shock driven test facility,"

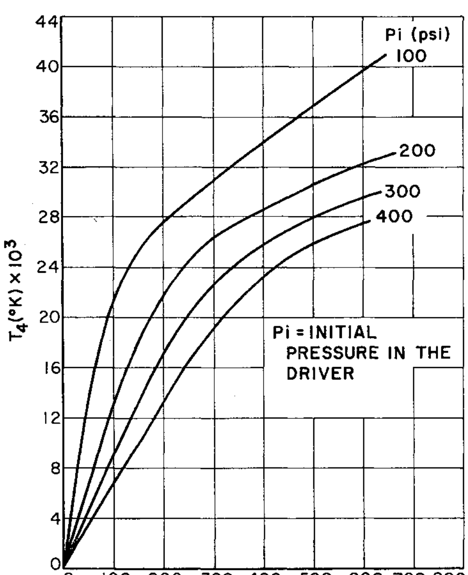


**Fig. 2 Resultant temperature for an energy input to a constant volume of helium**

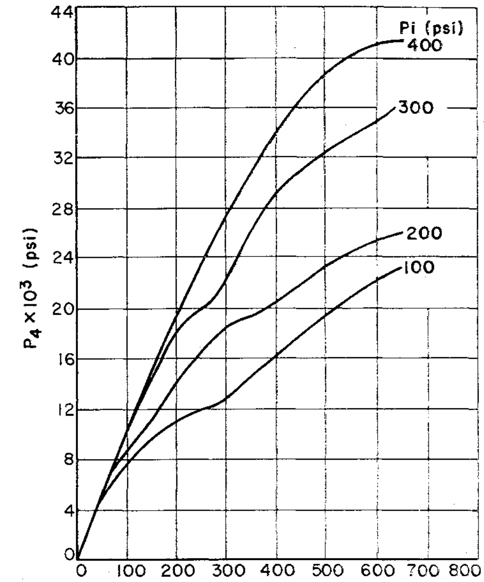


**Fig. 3 Resultant pressure for an energy input to a constant volume of helium**

Missiles and Space Vehicle Div., General Electric Co. Tech. Information Systems Rept. R62SD37 (April 1962).

[3] Lick, W. J. and Emmons, H. W., *Thermodynamic Properties of Helium to 50,000°K* (Harvard University Press, Cambridge, Mass., 1962).

---

# Discrete Element Approach to Structural Instability Analysis

Richard H. Gallagher* and Joseph Padlog†
*Textron's Bell Aerosystems Company, Buffalo, N. Y.*

Reference 1 suggests a matrix displacement formulation of the beam-column problem involving an assumption of linear displacements between element end points in the calculation of the axial load effects. The present note derives a general procedure for including instability effects in element force-displacement relationships applicable to both beams and plates and demonstrates that it can be used to obtain results to a high degree of accuracy with relatively few node points.

In the conventional approach to matrix displacement analysis, structural systems are idealized as assemblages of discrete elements. Relationships between the element juncture point forces $\{F\}$ and displacements $\{\delta\}$ are stated in the form $\{F\} = [k]\{\delta\}$, where $[k]$ is the element stiffness matrix. Upon evaluation, the element relationships are combined, in accordance with the requirements of node point equilibrium and compatible displacements, to yield

$$\{P\} = [K]\{\Delta\} \tag{1}$$

where $\{P\}$ are the node point external loads, $[K]$ is the stiffness matrix of the assembled idealization, and $\{\Delta\}$ are the node point displacements. The inverse of $[K]$ (after it has been modified in recognition of the geometric boundary conditions) is the set of structural influence coefficients. This note treats cases where Eq. (1) relates only the flexural forces and displacements, i.e., the midplane stress system is deter-

Received March 8, 1963.

* Chief, Advanced Airframe Analysis. Associate Fellow Member AIAA.

† Structures Research Engineer. Associate Member AIAA.

mined independently. The extension to the more general case can be accomplished through use of well-known concepts.

As shown in Refs. 2 and 3, Castigliano's first theorem, part I, provides the following matrix statement of a procedure for element stiffness matrix derivation in the absence of column effects:

$$[k] = ([B]^{-1})^T[C][B]^{-1} \tag{2}$$

where $[B]$ represents the set of relationships between the element node point displacements $\{\delta\}$ and the coefficients $\{a\}$ of functions chosen in approximation of the element displacement behavior, i.e., $\{\delta\} = [B]\{a\}$. The terms of $[B]$ merely are geometric in nature and do not relate explicitly to the phenomena under study. In the $[C]$ matrix, each row represents the coefficients in an equation for the derivative of the strain energy $(U)$, with respect to one of the coefficients $a$, where $U$ is expressed in terms of the element displacements and therefore in terms of the $a$'s.

For the inclusion of instability effects, it is necessary only to extend the element stiffness matrix $[k]$, since the assembly procedure for the master stiffness matrix $[K]$ is unaffected. To extend Eq. (2) to a form suitable for the development of stiffness matrices for flexural behavior in the presence of a known midplane force system, one first must examine fundamental considerations. If a structural element is subjected to a system of applied forces, $F_1, F_2, \ldots F_i \ldots F_N$, but regarded as fixed against all displacements $(u)$ except that which occurs at and in the direction of the single force $F_i$, it follows from the definition of work $(W)$ that

$$F_i = \partial W/\partial u_i \tag{3}$$

Since $W = U$ in the absence of column effects, $U$ being the corresponding strain energy of deformation, this becomes

$$F_i = \partial U/\partial u_i \tag{4}$$

which is an explicit statement of Castigliano's first theorem, part I, and the basis for Eq. (2).

For beams and plates under the conditions of interest, the following relationships exist between the work $W$ done by the lateral and midplane loads during bending deformation and the strain energy $U_f$ of bending (see Ref. 4):

$$W = W_v + W_h = U_f \tag{5}$$

where

$W_v$ = work done by the lateral loads $F_1, F_2, \ldots F_i \ldots F_N$ during flexure

$W_h$ = work done by the midplane loads during the displacement of the structure caused by the lateral loads $F_1, F_2, \ldots F_i \ldots F_N$

Furthermore, the change in $W_v$ with respect to the displacement $u_i$ can be obtained directly from Eq. (4) as

$$\partial W_v/\partial u_i = \partial(U_f - W_h)/\partial u_i \tag{6}$$

Now, if the restraint condition employed in the formulation of Eq. (3) is considered, it follows that

$$\partial W_v/\partial u_i = \partial W/\partial u_i \tag{7}$$

Consequently, from Eqs. (3, 5, and 6),

$$F_i = \partial(U_f - W_h)/\partial u_i \tag{8}$$

The transformation of Eq. (8) into a matrix formulation of the desired element stiffness matrix could be accomplished rigorously through application of the same procedures that led to Eq. (2). For brevity, however, the already-developed Eq. (2) will be used as a basis. First, it must be noted that $W_h$ can be expressed in terms of the lateral displacements. This is also the form taken by $U_f$. Hence, $(U_f - W_h)$ can be defined as an "effective" strain energy $U'$, and Eq. (7) can be written as

$$F_i = \partial U'/\partial u_i \tag{9}$$

Equation (4) is of similar appearance to Eq. (8) and was employed in the formulation of Eq. (2). To use Eq. (7) in the same way, one must note that, of the two matrices making up Eq. (2), only the matrix $[C]$ pertains to strain energy; each row of $[C]$ represents an equation for the derivative of the strain energy. Since now the strain energy $U'$ is composed of two parts, one can write

$$[C] = [C_f] - [C_h] \tag{10}$$

in which $[C_f]$ and $[C_h]$ result from the required operations on $U_f$ and $W_h$, respectively. Equation (2) therefore becomes

$$[k] = [k_f] - [k_h] = ([B]^{-1})^T[C_f][B]^{-1} - ([B]^{-1})^T[C_h][B]^{-1} \tag{11}$$

and Eq. (1) becomes

$$\{P\} = [[K_f] - [K_h]]\{\Delta\} \tag{12}$$

where $[K_f]$ and $[K_h]$ are assemblies of the $[k_f]$ and $[k_h]$ matrices, respectively. $[K_h]$ can be construed as an "incremental stiffness" of the system.

An equilibrium solution is achieved through inversion of the indicated total stiffness. For instability analysis, $\{P\} = 0$ and $[K_h]$ can be multiplied by the scalar $\lambda$, which then becomes the eigenvalue to be determined ($\lambda \leqslant 1.0$ for instability). Thus

$$(1/\lambda)\{\Delta_\lambda\} = [K_f]^{-1}[K_h]\{\Delta_\lambda\} \tag{13}$$

For uniform beam segments of length $l$,

$$U_f = \frac{EI}{2}\int_0^l \left(\frac{d^2w}{dx^2}\right)^2 dx \qquad W_h = -\frac{F_x}{2}\int_0^l \left(\frac{dw}{dx}\right)^2 dx \tag{14}$$

($F_x$, when tensile, is positive.)

For plates of constant flexural rigidity,

$$U_f = \frac{D}{2}\int_A \left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-u) \times \left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dA$$

$$W_h = -\frac{1}{2}\int_A \left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy} \times \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right] dA \tag{15}$$

($N_x$ and $N_y$, when tensile, are positive.)

To illustrate this approach, the beam segment of Fig. 1 is examined. With use of the assumed displacement function,

$$w = a_1 + a_2x + a_3x^2 + a_4x^3 \tag{16}$$

which is the exact shape for simple flexure, one has

$$\begin{Bmatrix} M_{y1} \\ M_{y2} \\ F_{z1} \\ F_{z2} \end{Bmatrix} = \left[\frac{2EI}{l^3}\begin{bmatrix} 2l^2 & l^2 & -3l & 3l \\ l^2 & 2l^2 & -3l & 3l \\ -3l & -3l & 6 & -6 \\ 3l & 3l & -6 & 6 \end{bmatrix} + \frac{F_x}{10l}\begin{bmatrix} \frac{4}{3}l^2 & -\frac{1}{3}l^2 & -l & l \\ -\frac{1}{3}l^2 & \frac{4}{3}l^2 & -l & l \\ -l & -l & 12 & -12 \\ l & l & -12 & 12 \end{bmatrix}\right]\begin{Bmatrix} \theta_{y1} \\ \theta_{y2} \\ w_1 \\ w_2 \end{Bmatrix} \tag{17}$$

The effect of transverse shear deformation can be included readily in the $[k_f]$ portion of the foregoing expression but for simplicity has been disregarded.

If symmetry is considered and one element is employed in the prediction of simple beam instability, one has $w_1 = \theta_{y2} =$

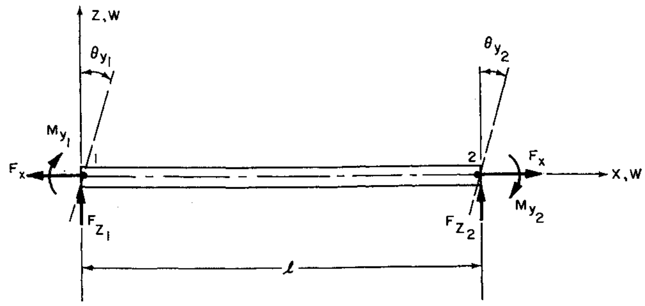


**Fig. 1 Uniform beam segment**

0, $F_x = -F_{x_{cr}}$, and $l = L/2$ in Eqs. (17), leading to the characteristic equation

$$\left(\frac{8EI}{L} - \frac{F_{x_{cr}}L}{15}\right)\left(\frac{96EI}{L^3} - \frac{12F_{x_{cr}}}{5L}\right) - \left(\frac{24EI}{L^2} - \frac{F_{x_{cr}}}{10}\right)^2 = 0 \quad (18)$$

Solving, one has $F_{x_{cr}} = 9.94(EI/L^2)$, which is only 0.752% different from the exact result, $\pi^2(EI/L^2)$. Reference 1 demonstrates a 10% error in a three-element solution.

It is possible to formulate the exact $[k_f] - [k_h]$ matrix through use of the proper beam column shape and Eq. (11) or by other means. The individual terms, being complicated functions of sines and cosines (or sinh and cosh), are not evaluated as easily as the terms derived in the foregoing. Furthermore, the foregoing technique applies equally well to plate elements in flexure for which exact displacement shapes cannot be found.

### References

[1] Rodden, W. P., Jones, J. P., and Bhuta, P. G., "A matrix formulation of the transverse structural influence coefficients of an axially loaded Timoshenko beam," AIAA J. 1, 225–227 (1963).

[2] Leissa, A. and Neidenfuhr, F. W., "A numerical study of the effects of various suggested modifications to the Turner program upon the Bell thick-skinned model wing," North American Aviation Rept. NA59A-670 (November 1959).

[3] Gallagher, R. H., "A correlation study of methods of matrix structural analysis," AGARDograph 69 (July 1962).

[4] Timoshenko, S., *Theory of Elastic Stability* (McGraw-Hill Book Co. Inc., New York, 1936), pp. 308–314.

# Minimum Structural Mass for a Magnetic Radiation Shield

S. W. Kash*
*Armour Research Foundation, Chicago, Ill.*

STRONG magnetic fields produced by superconducting coils have been suggested as a means of providing shielding in space against intense charged-particle radiation.[1–3] Estimates have shown that the mass of a magnetic shield will be less than that of a bulk shield designed to protect an equivalent volume of the vehicle against energetic protons of about 1 bev or more. These estimates have shown also that most of the mass in a magnetic shield will be contained in the structure required to support the superconducting coil against the large forces exerted upon it by its own magnetic field. It will be shown that a large reduction in the mass can be obtained by suitable design of the supporting structure. General limitations on the minimum structural mass associated with intense field producing coils also will be given.

Specific formulas will be developed for the structural mass required to support a circular cylindrical current sheet of arbitrary length $L$ and radius $R$. This model is a good approximation for a solenoidal coil for which the thickness of the windings is small in comparison with the radius. The results of this model for $L \gg R$ also are applicable directly to a torus, the minor radius of which is small compared with its major radius. Similarly, the results for $L \ll R$ can be applied to a coil consisting of a single circular turn.

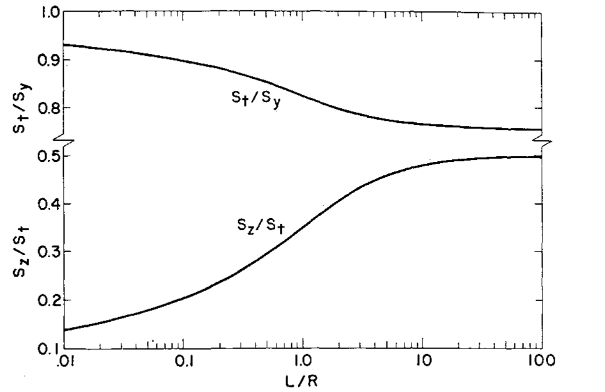


**Fig. 1 Variation of stress ratios $S_z/S_t$ and $S_t/S_y$ with $L/R$**

### Single Cylindrical Structure

Suppose that for support the solenoid is encased in a simple cylindrical structure $C_1$, the thickness $t$ of which is small compared with $R$. The force on each element of current in the solenoid can be resolved into a component directed radially outward and a component directed toward the equatorial plane of the solenoid. Consequently, the forces on the structure can be resolved into an (average) radial pressure $P$ tending to expand the structure and an axial force $F_z$ tending to compress the structure. The radial pressure will be balanced by an (average) azimuthal tensile stress $S_t$ in the structure. Thus, one has approximately

$$RP = tS_t \quad (1)$$

The axial force $F_z$ will be balanced by an (average) compressive axial stress $S_z$ in the structure; hence

$$F_z = 2\pi R t S_z \quad (2)$$

There also will be a radial compressive stress $S_r$ in the struc-

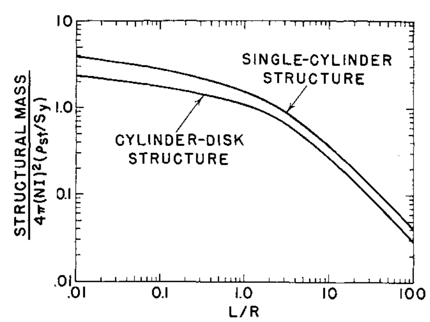


**Fig. 2 Variation of structural mass with coil shape $L/R$**

Received March 8, 1963. The author would like to express his appreciation to R. F. Tooper for a number of useful discussions concerning this problem and to R. J. Boden for assistance with the computations. This work was supported in part by the Air Force Aeronautical Systems Division.

* Manager, Plasma and Electron Physics Research, Physics Division. Associate Fellow Member AIAA.